ME451: Control Systems

Lecture 22
Frequency response

Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Course roadmap

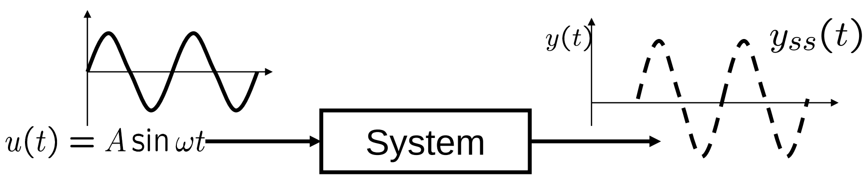
Modeling
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - (3) Bode plot
- Stability
  - Routh-Hurwitz
  - (3) Nyquist

Design
- (1) Design specs
- Root locus
- (3) Frequency domain
- (1) PID & Lead-lag
- (2) Design examples

(Matlab simulations &) laboratories

What is frequency response?

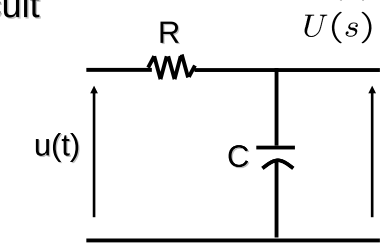


- We would like to analyze a system property by applying a *test sinusoidal input* u(t) and observing a response y(t).
- Steady state response $y_{ss}(t)$ (after transient dies out) of a system to sinusoidal inputs is called ***frequency response***.

A simple example

- RC circuit

$$Y(s) = \frac{1}{sC}I(s)$$
$$U(s) = (R + \frac{1}{sC})I(s)$$



$$G(s) = \frac{1}{RCs+1}$$

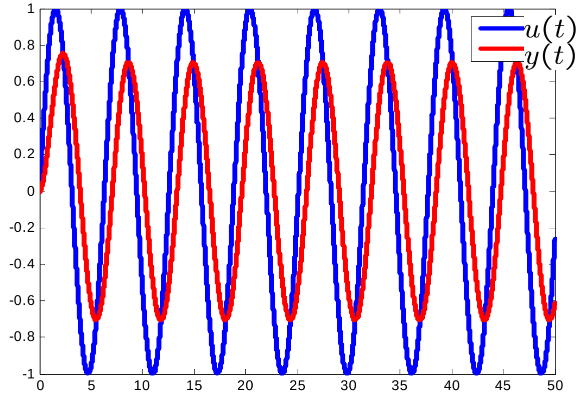
- Input a sinusoidal voltage u(t)
- What is the output voltage y(t)?

An example (cont'd)

- TF (R=C=1)

$$G(s) = \frac{1}{s+1}$$

- u(t)=sin(t)



At steady-state, u(t) and y(t) has same frequency, but different amplitude and phase!

An example (cont'd)

- Derivation of y(t)

$$Y(s) = G(s)U(s) = \frac{1}{s+1}\cdot\frac{1}{s^2+1} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right)$$

- Inverse Laplace

Partial fraction expansion

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

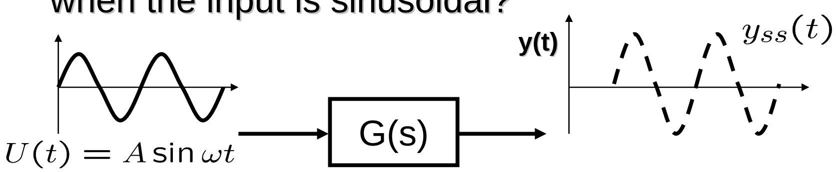
0 as t goes to infinity.

$$\Rightarrow \quad y_{ss}(t) = \frac{1}{2}(-\cos t + \sin t) = \frac{1}{\sqrt{2}}\sin(t - 45^\circ)$$

(Derivation for general G(s) is given at the end of lecture slide.)

Response to sinusoidal input

- How is the steady state output of a linear system when the input is sinusoidal?

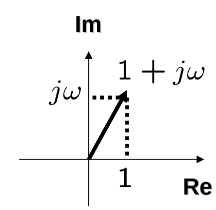


- Steady state output $y_{ss}(t) = A|G(j\omega)|\sin(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency ω
  - Amplitude is that of input (A) multiplied by $|G(j\omega)|$ Gain
  - Phase shifts $\angle G(j\omega)$

Frequency response function

- For a stable system G(s), G(jω) (ω is positive) is called *frequency response function (FRF)*.
- FRF is a complex number, and thus, has an amplitude and a phase.
- First order example

$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega + 1}$$



$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} \\ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \end{cases}$$

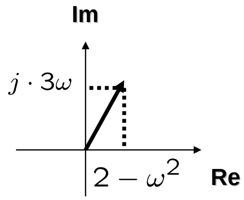
Another example of FRF

- Second order system

$$G(s) = \frac{2}{s^2 + 3s + 2}$$

$$\Rightarrow \quad G(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{2}{2 - \omega^2 + j \cdot 3\omega}$$

$$\Rightarrow \quad \begin{cases} |G(j\omega)| = \dfrac{2}{\sqrt{(2-\omega^2)^2 + 9\omega^2}} \\ \angle G(j\omega) = \angle(2) - \angle(2 - \omega^2 + j \cdot 3\omega) \\ \qquad\quad\; = -\tan^{-1} \dfrac{3\omega}{2 - \omega^2} \end{cases}$$



First order example revisited

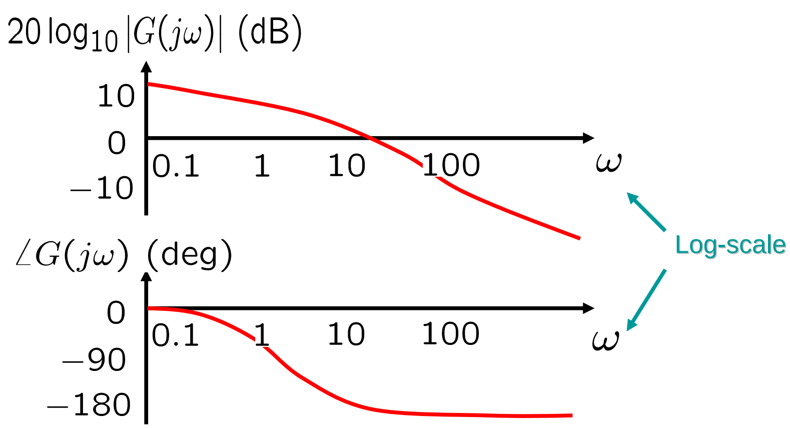
- FRF $G(j\omega) = \dfrac{1}{j\omega + 1}$

frequency ω	amplitude $\lvert G(j\omega)\rvert$	phase $\angle G(j\omega)$
0	1	$0°$
0.5	0.894	$-26.6°$
1.0	0.707	$-45°$
⋮	⋮	⋮
∞	0	$-90°$

- Two graphs representing FRF
  - Bode diagram (Bode plot) (Today)
  - Nyquist diagram (Nyquist plot)

Bode diagram (Bode plot) of G(jω)

- Bode diagram consists of gain plot & phase plot



Bode plot of a 1st order system

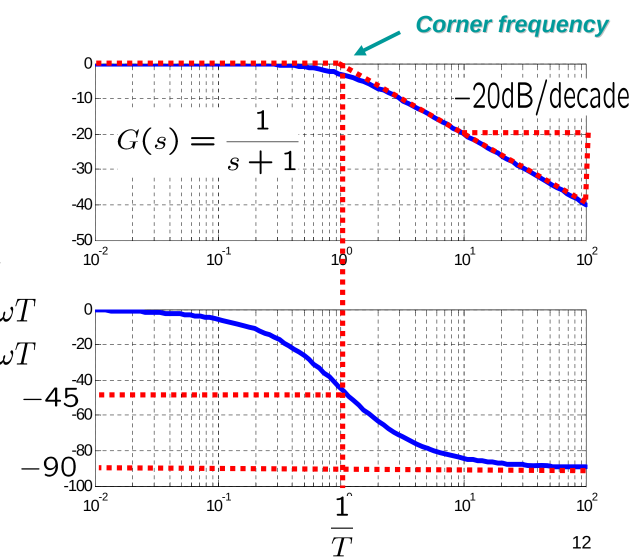
- TF

$$G(s) = \frac{1}{Ts + 1}$$

$$\Downarrow$$

$$G(j\omega) = \frac{1}{j\omega T + 1}$$

$$\approx \begin{cases} 1 & \text{if } 1 \gg \omega T \\ \dfrac{1}{j\omega T} & \text{if } 1 \ll \omega T \end{cases}$$



Exercises of sketching Bode plot

- First order system

$$G(s) = \frac{1}{s+1} \qquad G(s) = \frac{1}{0.1s+1} \qquad G(s) = \frac{1}{10s+1}$$

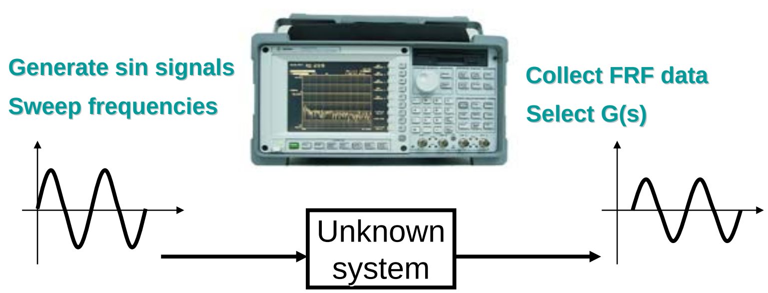
Remarks on Bode diagram

- Bode diagram shows amplification and phase shift of a system output for sinusoidal inputs with various frequencies.
- It is very useful and important in analysis and design of control systems.
- The shape of Bode plot contains information of stability, time responses, and much more!
- It can also be used for system identification. (Given FRF experimental data, obtain a transfer function that matches the data.)

System identification

- Sweep frequencies of sinusoidal signals and obtain FRF data (i.e., gain and phase).
- Select G(s) so that G(jω) fits the FRF data.

Agilent Technologies: FFT Dynamic Signal Analyzer



Summary and exercises

- Frequency response is a steady state response of systems to a sinusoidal input.
- For a linear system, sinusoidal input generates sinusoidal output with same frequency but different amplitude and phase.
- Bode plot is a graphical representation of frequency response function. ("bode.m")
- Next, Bode diagram of simple transfer functions
- Exercise: Read Section 8.

Derivation of frequency response

$$Y(s) = G(s)U(s) = G(s)\frac{A\omega}{s^2+\omega^2} = \frac{k_1}{s+j\omega} + \frac{k_2}{s-j\omega} + C_g(s)$$

Term having denominator of G(s)

$$\begin{cases} k_1 = \lim_{s\to -j\omega}(s+j\omega)G(s)\frac{A\omega}{s^2+\omega^2} = G(-j\omega)\frac{A\omega}{-2j\omega} = -\frac{AG(-j\omega)}{2j} \\ k_2 = \lim_{s\to j\omega}(s-j\omega)G(s)\frac{A\omega}{s^2+\omega^2} = G(j\omega)\frac{A\omega}{2j\omega} = \frac{AG(j\omega)}{2j} \end{cases}$$

0 as t goes to infinity.

$$\Rightarrow \quad y(t) = k_1 e^{-j\omega t} + k_2 e^{j\omega t} + \mathcal{L}^{-1}\{C_g(s)\}$$

$$\Rightarrow \quad y_{ss}(t) = A|G(j\omega)|\underbrace{\frac{e^{j(\omega t + \angle G(j\omega))} - e^{-j(\omega t + \angle G(j\omega))}}{2j}}_{\sin(\omega t + \angle G(j\omega))}$$