ME451: Control Systems

Lecture 18
Root locus: Sketch of proofs

Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Course roadmap

Modeling
- ✔ Laplace transform
- ✔ Transfer function
- ✔ Models for systems
  - ✔ electrical
  - ✔ mechanical
  - ✔ electromechanical
- ✔ Block diagrams
- ✔ Linearization

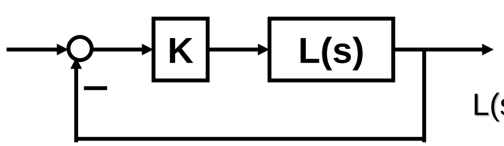
Analysis
- ✔ Time response
  - ✔ Transient
  - ✔ Steady state
- Frequency response
  - Bode plot
- ✔ Stability
  - ✔ Routh-Hurwitz
  - Nyquist

Design
- Design specs
- ➤ Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations &) laboratories

What is Root Locus? (Review)

- Consider a feedback system that has one parameter (gain) K>0 to be designed.



L(s): open-loop TF

- *Root locus* graphically shows how poles of the closed-loop system varies as K varies from 0 to infinity.

Characteristic equation & root locus

- Characteristic equation

$$1 + KL(s) = 0 \iff K = -\frac{1}{L(s)} \iff L(s) = -\frac{1}{K}$$

- Root locus is obtained by
  - for a fixed K>0, finding roots of the characteristic equation, and
  - sweeping K over real positive numbers.
- A point s is on the root locus, if and only if L(s) evaluated for that s is a negative real number.

Angle and magnitude conditions

- Characteristic eq. can be split into two conditions.
  - **Angle condition**

Odd number

$$\angle L(s) = 180^o \times (2k+1), \ k = 0, \pm 1, \pm 2, \ldots$$

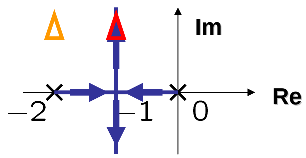
  - **Magnitude condition**

$$|L(s)| = \frac{1}{K}$$

For any point s, this condition holds for some positive K.

A simple example

$$L(s) = \frac{1}{s(s+2)}$$



- Select a point s=-1+j

$$L(s) = \frac{1}{s(s+2)} = \frac{1}{(-1+j)(1+j)} = -\frac{1}{2}$$

➡ $\angle L(s) = 180$

➡ s is on root locus.

$$K = \frac{1}{|L(s)|} = 2$$

- Select a point s=-2+j

$$L(s) = \frac{1}{s(s+2)} = \frac{1}{(-2+j)j} = \frac{1}{-2j-1}$$

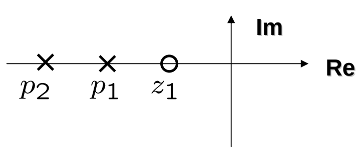
➡ $\angle L(s) \neq 180$

➡ s is NOT on root locus.

Root locus: Step 0

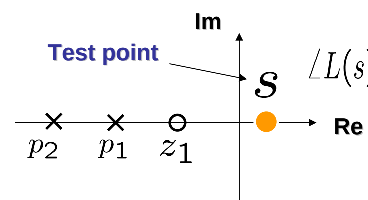
- *Root locus is symmetric w.r.t. the real axis.*
  - Characteristic equation is an equation with real coefficients. Hence, if a complex number is a root, its complex conjugate is also a root.
- *The number of branches = order of L(s)*
  - If L(s)=n(s)/d(s), then Ch. eq. is d(s)+Kn(s)=0, which has roots as many as the order of d(s).
- *Mark poles of L with "x" and zeros of L with "o".*

$$L(s) = \frac{s - z_1}{(s-p_1)(s-p_2)}$$



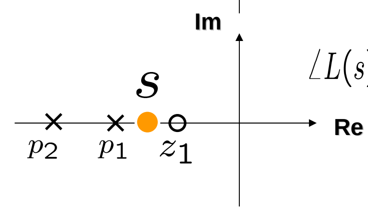
Root locus: Step 1-1

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*



$$\angle L(s) = \underbrace{\angle(s-z_1)}_{0} - \underbrace{\angle(s-p_1)}_{0} - \underbrace{\angle(s-p_2)}_{0}$$

Not satisfy angle condition!

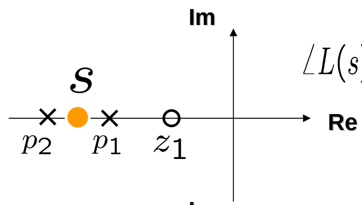


$$\angle L(s) = \underbrace{\angle(s-z_1)}_{180} - \underbrace{\angle(s-p_1)}_{0} - \underbrace{\angle(s-p_2)}_{0}$$

Satisfy angle condition!

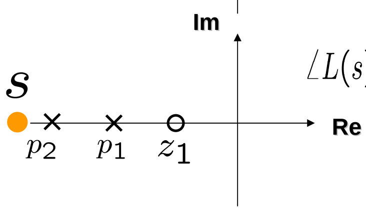
Root locus: Step 1-1 (cont'd)

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*



$$\angle L(s) = \underbrace{\angle(s-z_1)}_{180} - \underbrace{\angle(s-p_1)}_{180} - \underbrace{\angle(s-p_2)}_{0}$$

Not satisfy angle condition!



$$\angle L(s) = \underbrace{\angle(s-z_1)}_{180} - \underbrace{\angle(s-p_1)}_{180} - \underbrace{\angle(s-p_2)}_{180}$$

Satisfy angle condition!

Root locus: Step 1-2

- *RL originates from the poles of L, and terminates at the zeros of L, including infinity zeros.*

$$1+K\underbrace{\frac{n(s)}{d(s)}}_{L(s)} = 0 \Leftrightarrow d(s)+Kn(s) = 0 \Leftrightarrow \frac{1}{K}+\frac{n(s)}{d(s)} = 0$$

$K = 0$ ⇓ $d(s) = 0$ — **s: Poles of L(s)**

$K = \infty$ ⇓ $\frac{n(s)}{d(s)} = 0$ — **s: Zeros of L(s)**

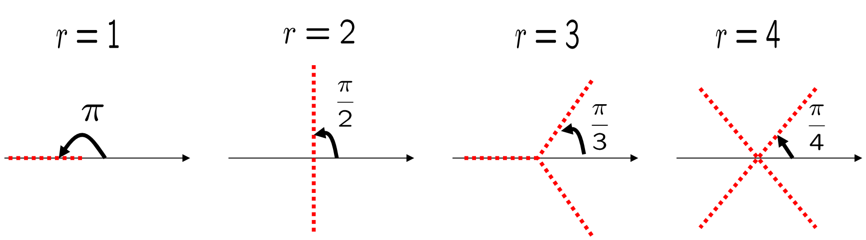
Root locus: Step 2-1

- *Number of asymptotes = relative degree (r) of L:*

$$r := \deg(\text{den}) - \deg(\text{num})$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1),\ k = 0, 1, \ldots$$



Root locus: Step 2-1 (cont'd)

- For a very large s, $L(s) = \frac{n_0 s^{n-r} + \cdots}{s^n + \cdots} \approx \frac{n_0}{s^r}$
- Ch. eq is approximately

$$1+KL(s) = 0 \Rightarrow 1+K\frac{n_0}{s^r} = 0 \Rightarrow s^r + Kn_0 = 0$$

$$\Rightarrow s^r = -Kn_0 < 0 \ (\text{we assume } n_0 > 0)$$

$$\Rightarrow \angle s^r = \pi \times (2k+1),\ k = 0, 1, 2, \ldots$$

$$\Rightarrow \angle s = \frac{\pi}{r} \times (2k+1),\ k = 0, 1, 2, \ldots$$

Root locus: Step 2-2

- *Intersections of asymptotes*

$$\frac{\sum \text{pole} - \sum \text{zero}}{r}$$

- Proof for this is omitted and not required in this course.
- Interested students should read page 363 in the book by Dorf & Bishop.

Root locus: Step 3

- *Breakaway points are among roots of* $\boxed{\frac{dL(s)}{ds} = 0}$

Suppose that s=b is a breakaway point.

$$\Rightarrow \begin{cases} d(b) + Kn(b) = 0 \\ d'(b) + Kn'(b) = 0 \end{cases} \Rightarrow d'(b) - \frac{d(b)}{n(b)} n'(b) = 0$$

$$\Rightarrow \left.\frac{dL(s)}{ds}\right|_{s=b} = \frac{n'(b)d(b) - n(b)d'(b)}{d(b)^2}$$

$$= -\frac{n(b)}{d^2(b)} \left\{ d'(b) - \frac{d(b)}{n(b)} n'(b) \right\} = 0$$

Root locus: Step 4

- RL departs from a pole p_j with angle of departure

$$\theta_d = \sum_i (p_j - z_i) - \sum_{i, i \neq j} (p_j - p_i) + 180$$

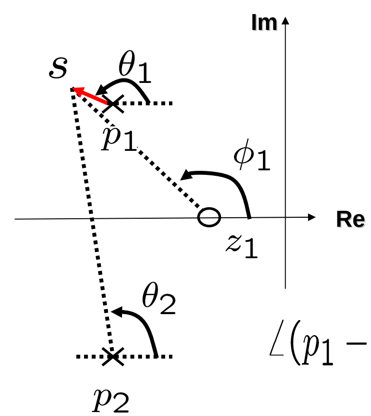
- RL arrives at a zero z_j with angle of arrival

$$\theta_a = \sum_i (z_j - p_i) - \sum_{i, i \neq j} (z_j - z_i) + 180$$

(No need to memorize these formula.)

Root locus: Step 4 (cont'd)

- Sketch of proof for angle of departure



For s to be on root locus, due to ***angle condition***

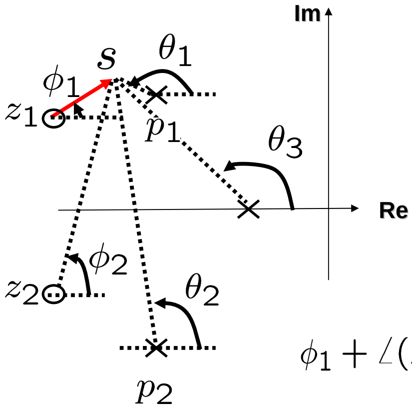
$$\phi_1 - \theta_1 - \theta_2 = 180$$

$$\Downarrow \quad |s - p_1| \to 0$$

$$\angle(p_1 - z_1) - \theta_1 - \angle(p_1 - p_2) = 180$$

Root locus: Step 4 (cont'd)

- Sketch of proof for angle of arrival



For s to be on root locus, due to ***angle condition***

$$\phi_1 + \phi_2 - \theta_1 - \theta_2 - \theta_3 = 180$$

$$|s - z_1| \to 0$$

$$\phi_1 + \angle(z_1 - z_2) - \sum_{i=1}^{3} \angle(z_1 - p_i) = 180$$

Summary and exercises

- Sketch of proofs for root locus algorithm
- Next, we will move on to root locus applications to control examples.