

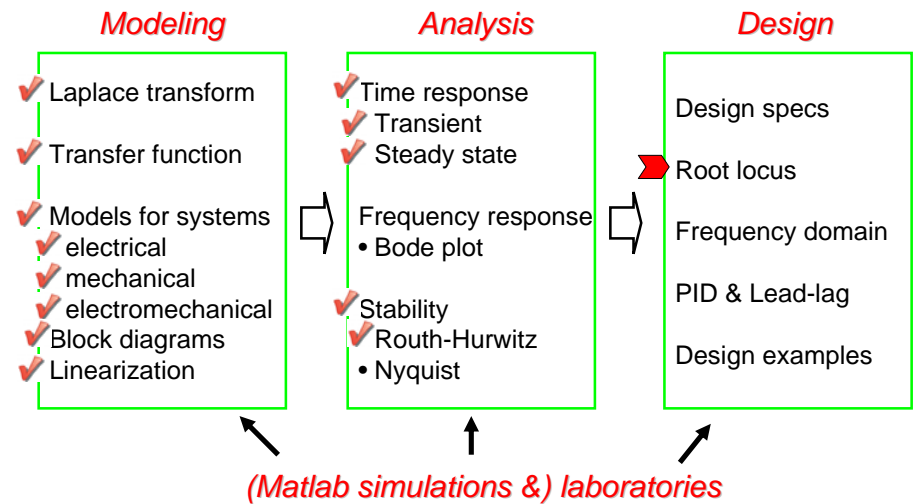
ME451: Control Systems

Lecture 17 Root locus: Examples

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1

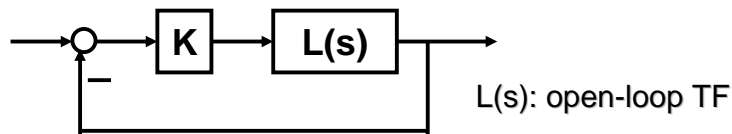
Course roadmap



2

What is Root Locus? (Review)

- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed.



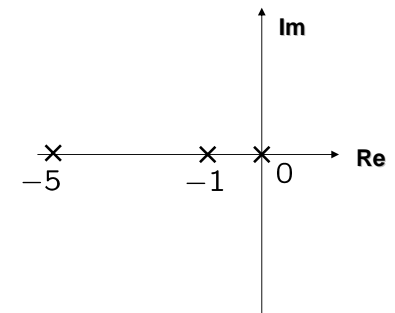
- Root locus** graphically shows how poles of the closed-loop system varies as K varies from 0 to infinity.

3

Root locus: Step 0 (Mark pole/zero)

- Root locus is symmetric w.r.t. the real axis.
- The number of branches = order of $L(s)$
- Mark poles of L with "x" and zeros of L with "o".

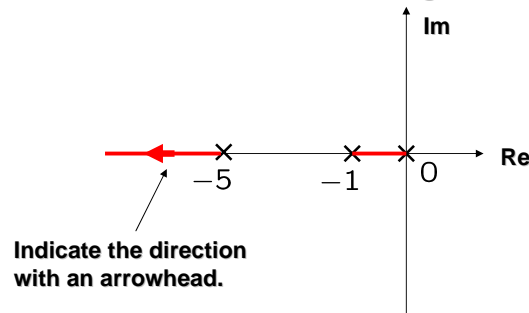
$$L(s) = \frac{1}{s(s+1)(s+5)}$$



4

Root locus: Step 1 (Real axis)

- RL includes all points on real axis to the left of an odd number of real poles/zeros.
- RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.

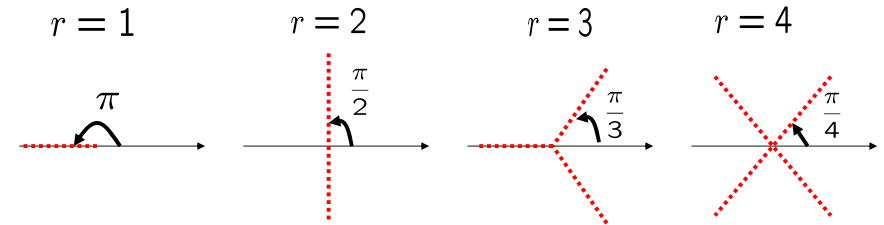


5

Root locus: Step 2 (Asymptotes)

- Number of asymptotes = relative degree (r) of L:
 $r := \deg(\text{den}) - \deg(\text{num})$
- Angles of asymptotes are

$$\frac{\pi}{r} \times (2k + 1), \quad k = 0, 1, \dots$$

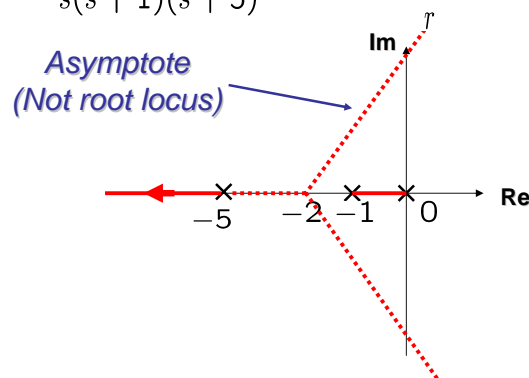


6

Root locus: Step 2 (Asymptotes)

- Intersections of asymptotes $\frac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \quad \rightarrow \quad \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{0 + (-1) + (-5)}{3} = -2$$



7

Root locus: Step 3 (Breakaway)

- Breakaway points are among roots of $\frac{dL(s)}{ds} = 0$

$$L(s) = \frac{1}{s(s+1)(s+5)} \quad \rightarrow \quad \frac{dL(s)}{ds} = -\frac{3s^2 + 12s + 5}{(*)} = 0$$

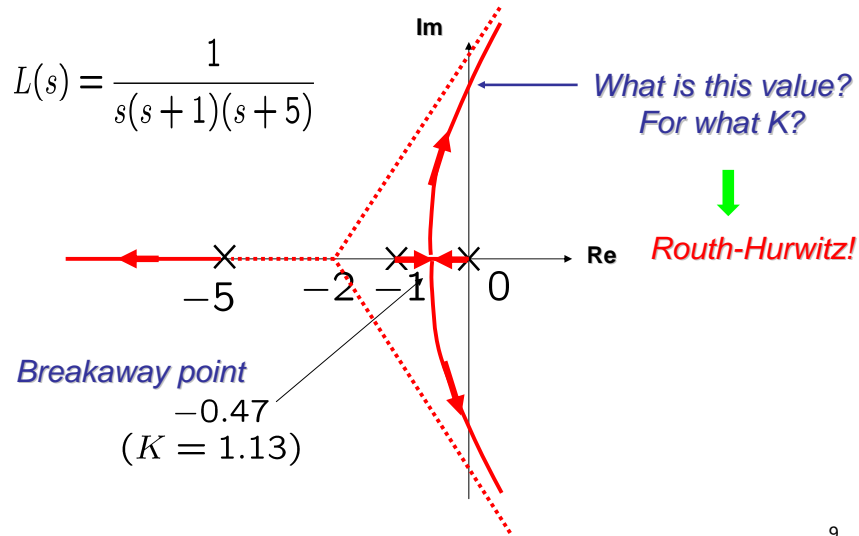
$$\rightarrow s = -2 \pm \frac{\sqrt{21}}{3}$$

For each candidate s, check the positivity of $K = -\frac{1}{L(s)}$

$$\rightarrow \begin{cases} s = -2 + \frac{\sqrt{21}}{3} \approx -0.47 & K \approx 1.13 \\ s = -2 - \frac{\sqrt{21}}{3} \approx -3.52 & K \approx -13.1 \end{cases}$$

8

Root locus: Step 3 (Breakaway)



9

Finding K for critical stability

- Characteristic equation

$$1 + \frac{K}{s(s+1)(s+5)} = 0 \Leftrightarrow s^3 + 6s^2 + 5s + K = 0$$

- Routh array

$$\begin{array}{c|cc}
 s^3 & 1 & 5 \\
 s^2 & 6 & K \\
 s^1 & \frac{30-K}{6} & \\
 s^0 & K &
 \end{array}$$

Stability condition

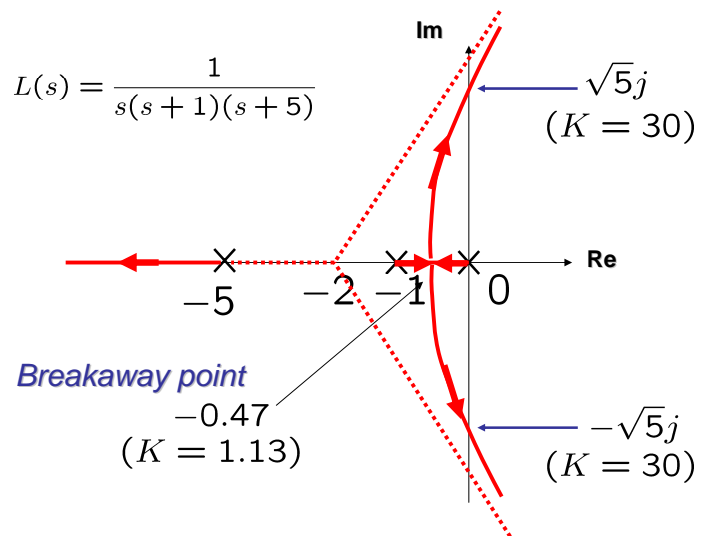
$$0 < K < 30$$

- When $K=30$

$$6s^2 + 30 = 0 \Rightarrow s = \pm\sqrt{5}j$$

10

Root locus



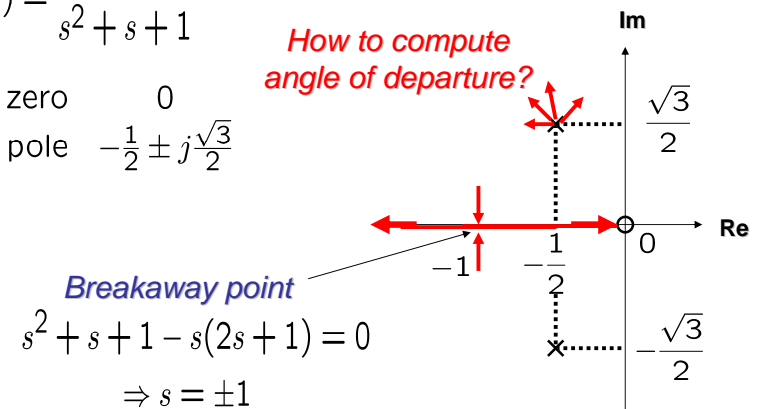
11

Example with complex poles

$$L(s) = \frac{s}{s^2 + s + 1}$$

zero 0
 pole $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

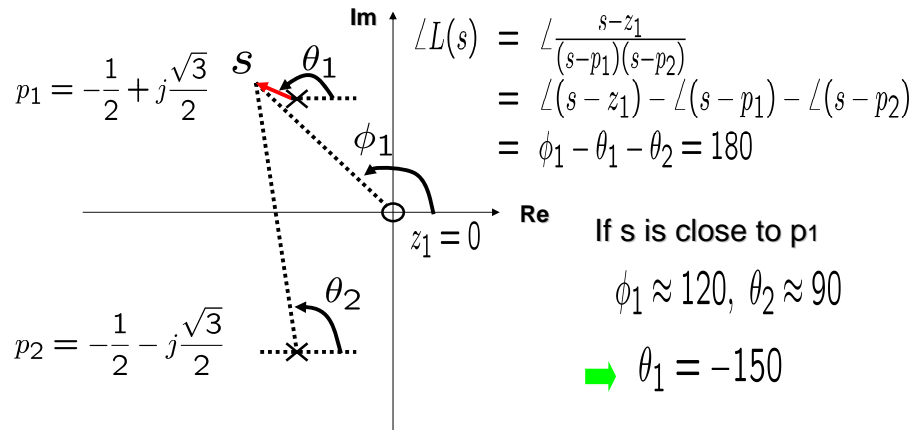
After Steps 0,1,2,3, we obtain



12

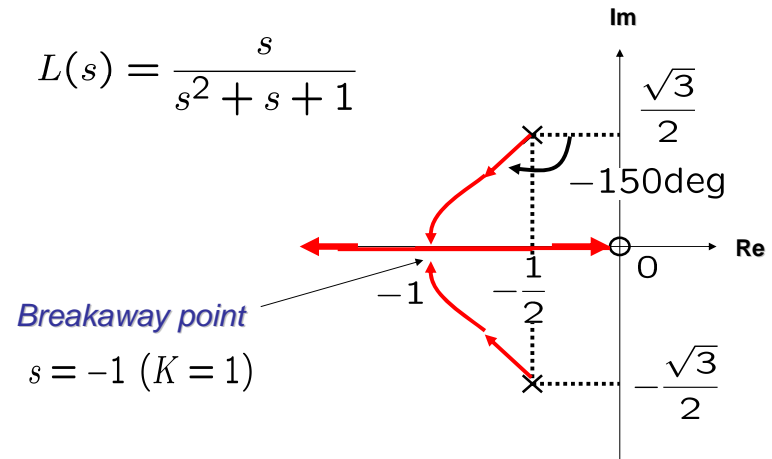
Root locus: Step 4 Angle of departure

- **Angle condition:** For s to be on RL,



13

Root locus



14

Summary and exercises

- Examples for root locus.
 - Gain computation for marginal stability, by using Routh-Hurwitz criterion
 - Angle of departure (Angle of arrival can be obtained by a similar argument.)
- Next, sketch of proofs for root locus algorithm
- Exercises
 - Draw root locus for $K > 0$ (no need to consider $K < 0$) for open-loop transfer functions in
 - Problems 7.5 and 7.7.

15

Exercises 1

$$L(s) = \frac{s}{s^2 + s + 1} \quad L(s) = \frac{1}{(s - 1)(s + 2)(s + 3)}$$

$$L(s) = \frac{s + 1}{s^2} \quad L(s) = \frac{s}{(s + 1)(s^2 + 1)}$$

16

Exercises 2

$$L(s) = \frac{1}{s(s+1)(s+2)}$$

$$L(s) = \frac{1}{(s+1)(s^2+2s+2)}$$

$$L(s) = \frac{1}{s(s+2)(s^2+2s+2)}$$

$$L(s) = \frac{1}{(s^2+4s+5)(s^2+2s+5)}$$

17

Exercises 3

$$L(s) = \frac{1}{s(s+3)(s^2+2s+2)}$$

$$L(s) = \frac{1}{s(s+1)(s+2)(s^2+2s+2)}$$

$$L(s) = \frac{s+1}{s^2+4s+5}$$

$$L(s) = \frac{s+3}{(s+1)(s^2+4s+5)}$$

18

Exercises 4

$$L(s) = \frac{s+4}{(s+1)(s+3)(s^2+4s+5)}$$

$$L(s) = \frac{s+2}{(s^2+2s+5)(s^2+6s+10)}$$

$$L(s) = \frac{s^2+2s+2}{s(s+2)(s+3)}$$

$$L(s) = \frac{(s+2)(s+3)}{s(s+1)}$$

19